Functional Quantum Theory of Free Relativistic Fermi Fields

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Functional quantum theory of free Fermi fields is treated for the special case of a free Dirac field. All other cases run on the same pattern. Starting with the Schwinger functionals of the free Dirac field, functional equations and corresponding many particle functionals can be derived. To establish a functional quantum theory, a physical interpretation of the functionals is required. It is provided by a mapping of the physical Hilbert space into an appropriate functional Hilbert space, which is introduced here. Mathematical details, especially the problems connected with anticommuting functional sources are treated in the appendices.

The operator equations of quantum field theory can be replaced formally by functional equations of corresponding Schwinger functionals [1–3]. To give this formalism a physical and mathematical meaning one has to develop a complete functional quantum theory as has been proposed in preceding papers [4, 5]. Then the complete physical information has to be given by functional operations only. In this paper this program is realized for free relativistic Fermi fields. This treatment is of physical as well as of mathematical interest. Mathematically it requires a thorough investigation of anticommuting sources, which has not been given so far. Physically it is required for the functional S-matrix construction of interacting fields, as commonly is assumed that these fields contain asymptotic free particle states which have to be described by functionals, too. In order to give a detailed treatment, we discuss a special case of free relativistic Fermi fields, namely the Dirac field. But the discussion shows that an application to other types of Fermi fields is quite obvious. To separate the essentials from mathematical details, the latter ones are treated in the appendices. Throughout this article $\hbar = c = 1$ is used.

1. Fundamentals

For the discussion of free relativistic Fermi fields we consider the special case of Dirac fields. Dirac fields can be formulated by Hermitean field operators [6]. By the use of these operators functional calculations are simplified and the structural content of the theory becomes transparent. A thorough discussion of them is given in Appendix I. Denoting a Hermitean Dirac spinor by $\Psi_\alpha(x)$ the field equations read

$$(i^\mu G_\alpha^\beta \, \partial_\mu + m \, \delta_\alpha^\beta) \; \Psi_\beta(x) = 0 \qquad (1.1)$$

where the quantities occurring in (1.1) are defined in Appendix I. The anticommutation relations for the quantized field are

$$[\Psi_\alpha(x) \; \Psi_\beta(x')]_{+/x_0 = x_0'} = \delta_{\alpha\beta} \, \delta(\mathfrak{r} - \mathfrak{r}') . \qquad (1.2)$$

For (1.1) (1.2) a representation space can be constructed containing all many particle states of an arbitrary number of free relativistic Dirac particles with mass m. We assume $|a\rangle$ to be a state in this space. If necessary $|a\rangle$ can be specified further by its set of quantum numbers.

Of fundamental interest are the transformation properties of the field operators. By invariance requirements they transform according the law

$$U \, \Psi_\beta(x) \; U^{-1} = D_\beta^\varkappa \, \Psi_\varkappa(a\,x + b) \qquad (1.3)$$

where (a, b) are the transformation operators of a Poincaré transformation in ordinary Lorentz space, D means a representation of (a, b) in classical spinor space, and U a representation of (a, b) in physical Hilbert space.

For the functional description we introduce spinorial source operators $j^\alpha(x)$ and corresponding operators of functional differentiation $\partial_\beta(x)$ satis-

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fying the anticommutation relations

$$[j^\alpha(x)\, \partial_\beta(x')]_+ = \delta^\alpha_\beta\, \delta(x-x'),$$
$$[j^\alpha(x)\, j^\beta(x')]_+ = 0, \qquad (1.4)$$
$$[\partial_\alpha(x)\, \partial_\beta(x')]_+ = 0.$$

The transformation properties of these operators are assumed to be

$$V\, j^\alpha(x)\, V^{-1} = D^{-1\alpha}_{\ \ \beta}\, j^\beta(a\,x+b),$$
$$V\, \partial_\alpha(x)\, V^{-1} = D^\beta_\alpha\, \partial_\beta(a\,x+b) \qquad (1.5)$$

where V means a representation of (a, b) in the corresponding functional space. A detailed discussion is given in Appendix II.

Then the generating Schwinger functional is defined by

$$\mathfrak{T}(j, a) := \langle 0 | T \exp\{i \int \Psi_\alpha(x)\, j^\alpha(x)\, \mathrm{d}x\} | a \rangle$$
$$:= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \langle 0 | T\, \Psi_{\alpha_1}(x_1) \dots \Psi_{\alpha_n}(x_n) | a \rangle \cdot j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n \qquad (1.6)$$

where $\langle 0 |$ is the physical vacuum state. Observing the transformation properties of fields (1.3) and sources (1.4) it follows that $\mathfrak{T}(j)$ transforms under Poincaré transformations like an invariant operator [7]

$$\mathfrak{T}'(j, a) = V\, \mathfrak{T}(j, a)\, V^{-1}. \qquad (1.7)$$

To define the functional representation space completely, a functional vacuum state $|\varphi_0\rangle$ has to be introduced. Then

$$|\mathfrak{T}(j, a)\rangle := \mathfrak{T}(j, a)\, |\varphi_0\rangle \qquad (1.8)$$

transforms like a functional state and for $|\mathfrak{T}(j, a)\rangle$ the functional equation

$$(i^\mu\, G^\beta_\alpha\, \partial_\mu + m\, \delta^\beta_\alpha)\, \partial_\beta(x)\, |\mathfrak{T}(j, a)\rangle = -i\, {}^0G^\beta_\alpha\, j_\beta(x)\, |\mathfrak{T}(j, a)\rangle \qquad (1.9)$$

can be derived, see Appendix III. It should be emphasized that this derivation is possible only for the states (1.8) but not for the operators. So the definition of the functional vacuum state is an important step.

2. Eigenstate Functionals

Defining by k the maximal set of quantum numbers for the quantum mechanical one particle solutions of the nonquantized Dirac equation, the base vectors of the fieldtheoretic physical Hilbert space are given by $|a\rangle = |k_1 \dots k_n\rangle$ where $n = 0, 1, \dots, \infty$ and k_a runs through all possible quantum number combinations of the maximal set. The corresponding functionals are defined by $|\mathfrak{T}(j, k_1 \dots k_n)\rangle$ which can be obtained from (1.6), (1.8) by substitution of a physical base vector for $|a\rangle$. One may suppose that these states play a similar role in functional space, i. e. that they define a proper base vector set in this space. The calculation of them is possible by direct evaluation of the definition (1.6), (1.8) for $|\mathfrak{T}(j, k_1 \dots k_n)\rangle$.

But a shorter procedure is the calculation by means of the functional equation (1.9). To do this we make the general ansatz

$$|\mathfrak{T}(j)\rangle = \exp\{-\tfrac{1}{2} \int j^\alpha(x)\, F_{\alpha\beta}(x-y)\, j^\beta(y)\, \mathrm{d}x\, \mathrm{d}y\}\, |\Phi(j)\rangle. \qquad (2.1)$$

Then by elementary transformation of (1.9) an equation for $|\Phi(j)\rangle$ can be derived:

$$(i^\mu\, G^\beta_\alpha\, \partial_\mu + m\, \delta^\beta_\alpha)\, [-\int F_{\beta\gamma}(x-y)\, j^\gamma(y)\, \mathrm{d}y + \partial_\beta(x)]\, |\Phi(j)\rangle = -i\, {}^0G^\beta_\alpha\, j_\beta(x)\, |\Phi(j)\rangle. \qquad (2.2)$$

Assuming now F to be a solution of

$$(i^\mu\, G^\beta_\alpha\, \partial_\mu + m\, \delta^\beta_\alpha)\, F_{\beta\gamma}(x-y) = -i\, {}^0G_{\alpha\gamma}\, \delta(x-y) \qquad (2.3)$$

for $|\Phi(j)\rangle$ the equation

$$(i^\mu\, G^\beta_\alpha\, \partial_\mu + m\, \delta^\beta_\alpha)\, \partial_\beta(x)\, |\Phi(j)\rangle = 0 \qquad (2.4)$$

follows. Denoting by $f_\beta(x|k)$ the quantum mechanical one particle solutions of the Dirac-equation (1.1) a solution of (2.4) is given by

$$|\Phi(j)\rangle = \frac{1}{n!}\, P \sum_{\lambda_1 \dots \lambda_n} \int (-1)^P f_{\beta_1}(x_1 | k_{\lambda_1}) \dots f_{\beta_n}(x_n | k_{\lambda_n})\, j^{\beta_1}(x_1) \dots j^{\beta_n}(x_n)\, |\varphi_0\rangle\, \mathrm{d}x_1 \dots \mathrm{d}x_n. \qquad (2.5)$$

[7] H. Stumpf, H. G. Märtl, and K. Scheerer, to be publ.

This can be verified by elementary calculus. As in (2.5) the quantum numbers $k_1 \dots k_n$ occur, it is obvious to put

$$|\mathfrak{T}(j, k_1 \dots k_n)\rangle = \exp\{-\tfrac{1}{2}\int j^{\alpha}(x)\, F_{\alpha\beta}(x-y)\, j^{\beta}(y)\, dx\, dy\} \times \frac{1}{n!} p \sum_{\lambda_1 \dots \lambda_n} \int (-1)^p f_{\beta_1}(x_1 | k_{\lambda_1}) \dots f_{\beta_n}(x_n | k_{\lambda_n})\, j^{\beta_1}(x_1) \dots j^{\beta_n}(x_n)\, |\varphi_0\rangle\, dx_1 \dots dx_n . \tag{2.6}$$

The final proof that (2.6) is valid can be given by means of subsidiary conditions. Due to the group theoretical structure of quantum theory any quantum number is connected with the diagonalization of an infinitesimal operator of the corresponding symmetry group. For these operators a representation in functional space can be given, too [7]. In consequence any quantum number definition in physical Hilbert space by means of infinitesimal operators can be transferred in functional space leading there to the so-called subsidiary conditions. As for this statements we refer to [7] we do not give here a detailed treatment. We only state that by means of these conditions (2.6) can be justified.

3. Functional Orthonormalization

In order to derive a functional quantum theory it is necessary to define a mapping of the physical Hilbert space into functional space, and to obtain by this way a physical interpretation of functional states. To do this we introduce a functional scalar product and examine its effect on state functionals of type (2.6). In analogy to the treatment of the harmonic oscillator functionals [8] we define the weighted functional scalar product by

$$\langle \overline{\mathfrak{T}(j)} | \mathfrak{T}(j) \rangle_W := \langle \overline{g(j)\, \mathfrak{T}(j)} | g(j)\, \mathfrak{T}(j) \rangle \tag{3.1}$$

with

$$g(j) := e^{-\frac{1}{2} j (G-F) j} \tag{3.2}$$

where F is the twopoint function of Sect. 2, while G is a general weighting function which will be fixed later. Observing (II.19), (II.34), and (2.6) we obtain from (3.1)

$$\langle \overline{\mathfrak{T}(j, k_1' \dots k_n')} | \mathfrak{T}(j, k_1 \dots k_m) \rangle_W = \int p' \sum_{\lambda_1' \dots \lambda_n'} (-1)^{p'} f_{\beta_1'}(x_1' | k_{\lambda_1'}') \dots f_{\beta_n'}(x_n' | k_{\lambda_n'}')\; p \sum_{\lambda_1 \dots \lambda_m} (-1)^p f_{\beta_1}(x_m | k_{\lambda_1}) \dots f_{\beta_m}(x_1 | k_{\lambda_m}) \times \langle \overline{D_n^{\beta_1' \dots \beta_n'}(x_1' \dots x_n')} | D_m^{\beta_1 \dots \beta_m}(x_1 \dots x_m) \rangle\, dx_1' \dots dx_n'\, dx_1 \dots dx_m . \tag{3.3}$$

As the functions $f_\beta(x|k)$ are real functions, they are reproduced by complex conjugation in the adjoint functional $\langle \overline{\mathfrak{T}(j)}|$ leading thus to (3.3). Using the summation convention of Appendix II evaluation of (3.3) acording to (II.45) gives

$$\langle \overline{\mathfrak{T}(j, k_1' \dots k_n')} | \mathfrak{T}(j, k_1 \dots k_m) \rangle_W = p' \sum_{\lambda_1' \dots \lambda_n'} (-1)^{p'} f(x_1' | k_{\lambda_1}') \dots f(x_n' | k_{\lambda_n}')\; p \sum_{\lambda_1 \dots \lambda_m} (-1)^p f(x_1 | k_{\lambda_1}) \dots f(x_m | k_{\lambda_m}) \times \sum_{\varrho} 2^{2\varrho-(n+m)/2} \frac{(-1)^{(n+m)/2}}{\frac{1}{2}(n-2\varrho)! \cdot \frac{1}{2}(m-2\varrho)!} \frac{1}{(2\varrho)!^2} \bar{\Delta}(x'^1 \xi'^1) \dots \bar{\Delta}(x'^n \xi'^n)\, \Delta(x^1 \xi^1) \dots \Delta(x^m \xi^m) \times A(\xi_1 \xi_1') \dots A(\xi_{2\varrho} \xi_{2\varrho}')\, C(\xi_{2\varrho+1}' \xi_{2\varrho+2}') \dots C(\xi_{n-1}' \xi_n')\, C(\xi_{2\varrho+1} \xi_{2\varrho+2}) \dots C(\xi_{m-1} \xi_m). \tag{3.4}$$

For the effective calculation of (3.4) it is necessary to fix the weighting function G. We make the ansatz

$$G_{\varkappa'}^{\varkappa}(x, x') := \delta(x-x')\, f_{\varkappa'}^{\varkappa}(x') \tag{3.5}$$

and

$$f_{\varkappa'}^{\varkappa}(x') := i^{\lambda} G_{\varkappa'}^{\varkappa}\, a_\lambda\, f(a^\lambda x_\lambda') \tag{3.6}$$

both suggested by the current expression of Appendix I. As (3.5), (3.6) are Poincaré forminvariant quantities, it is sufficient to evaluate (3.4) in a special frame of reference defined by $a := (1, 0, 0, 0)$. In this frame of reference (3.5) becomes

$$G_{\varkappa'}^{\varkappa}(x, x') = Z_{\varkappa'}^{\varkappa}\, \delta(x-x')\, f(a x') \tag{3.7}$$

[8] R. WEBER, Thesis, University Tübingen 1970.

with

$$Z^{\varkappa}_{\varkappa'} := i\,{}^0G^{\varkappa}_{\varkappa'}\,. \tag{3.8}$$

All weighting functions appearing in (3.4) are functionals of G. The corresponding functional relations are derived in Appendix II, namely (II.25), (II.32), (II.36), (II.39), and (II.43). By the special choice of G (3.5), (3.6), resp. (3.7), (3.8) a straightforward and exact calculation of these functions is possible. We give its result for the rest frame (3.7), (3.8)

$$\begin{aligned} \underset{\varkappa\varkappa'}{A}(x\,x') &= ({}^5G\,{}^0G)_{\varkappa\varkappa'}\,\delta(x-x')\,a(x'),\\ \underset{\varkappa\varkappa'}{C}(x\,x') &= Z_{\varkappa\varkappa'}\,\delta(x-x')\,c(x') = \underset{\varkappa\varkappa'}{\overline{C}}(x\,x'),\\ \underset{\varkappa\varkappa'}{\Delta}(x\,x') &= g_{\varkappa\varkappa'}\,\delta(x-x')\,\mathrm{d}(x') = \underset{\varkappa\varkappa'}{\overline{\Delta}}(x\,x') \end{aligned} \tag{3.9}$$

with

$$\begin{aligned} a(x') &:= 1+f^2(x'),\\ c(x') &:= f(x')\,a(x'),\\ \mathrm{d}(x') &:= \sum_{u=0}^{\infty} f^{2n}(x'). \end{aligned} \tag{3.10}$$

For (3.4) the following combinations are required

$$\int \overset{\beta'\eta'}{\Delta}(x'\,\xi')\,\underset{\eta'\eta}{A}(\xi'\,\xi)\,\overset{\eta\beta}{\Delta}(\xi\,x)\,\mathrm{d}\xi\,\mathrm{d}\xi' = ({}^5G\,{}^0G)^{\beta\beta'}\,\delta(x-x')\,b_1(x') \tag{3.11}$$

with

$$b_1(x') := d^2(x')\,a(x') \tag{3.12}$$

and

$$\int \overset{\beta'\eta'}{\Delta}(x'\,\xi')\,\underset{\eta'\eta}{C}(\xi'\,\xi)\,\overset{\eta\beta}{\Delta}(\xi\,x)\,\mathrm{d}\xi\,\mathrm{d}\xi' = Z^{\beta\beta'}\,\delta(x-x')\,b_2(x') \tag{3.13}$$

with

$$b_2(x') := d^2(x')\,f(x')\,a(x'). \tag{3.14}$$

Substitution of (3.11), (3.13) into (3.4) gives then

$$\begin{aligned} &\langle \overline{\mathfrak{T}(j,k_1' \ldots k_n')}\,|\,\mathfrak{T}(j,k_1 \ldots k_m)\rangle_W \\ &= \int p' \sum_{\lambda_1' \ldots \lambda_n'} (-1)^{p'} f_{\beta_1'}(x_1'\,|\,k_{\lambda_1'}') \ldots f_{\beta_n'}(x_n'\,|\,k_{\lambda_n'}')\; p \sum_{\lambda_1 \ldots \lambda_m} (-1)^p f_{\beta_1}(x_1\,|\,k_{\lambda_1}) \ldots f_{\beta_1}(x_m\,|\,k_{\lambda_m}) \\ &\times \sum_{\varrho} 2^{2\varrho-(n+m)/2} \frac{(-1)^{(n+m)/2}}{\frac{1}{2}(n-2\varrho)!\cdot\frac{1}{2}(m-2\varrho)!}\,\frac{1}{(2\varrho)!^2}\,({}^5G\,{}^0G)^{\beta_1'\beta_1}\,\delta(x_1'-x_1)\,b_1(x_2') \ldots \\ &\qquad ({}^5G\,{}^0G)^{\beta'_{2\varrho}\beta_{2\varrho}}\,\delta(x_{2\varrho}'-x_{2\varrho})\,b(x_{2\varrho}') \\ &\times Z^{\beta'_{2\varrho+1}\beta'_{2\varrho+2}}\,\delta(x_{2\varrho+1}'-x_{2\varrho+2}')\,b_2(x_{2\varrho+2}') \ldots Z^{\beta'_{n-1}\beta'_n}\,\delta(x_{n-1}'-x_n')\,b_2(x_n') \\ &\times Z^{\beta_{2\varrho+1}\beta_{2\varrho+2}}\,\delta(x_{2\varrho+1}-x_{2\varrho+2})\,b_2(x_{2\varrho+2}) \ldots Z^{\beta_{m-1}\beta_m}\,\delta(x_{m-1}-x_m)\,b_2(x_m)\,\mathrm{d}x_1' \ldots \mathrm{d}x_n'\,\mathrm{d}x_1 \ldots \mathrm{d}x_m\,. \end{aligned} \tag{3.15}$$

Now it is according to (I.39), (I.40)

$$\begin{aligned} \int f_{\beta'}(x\,|\,k)\,({}^5G\,{}^0G)^{\beta'\beta} f_\beta(x\,|\,k')\,b_1(x_0)\,\mathrm{d}x &= \int [f_{1\alpha} f_{1\alpha}' + f_{2\alpha} f_{2\alpha}']\,b_1(x_0)\,\mathrm{d}x = \delta_{kk'}\,,\\ \int f_{\beta'}(x\,|\,k)\;Z^{\beta'\beta} f_\beta(x\,|\,k')\,b_2(x_0)\,\mathrm{d}x &= \int [f_{1\alpha} f_{2\alpha}' - f_{2\alpha} f_{1\alpha}']\,b_2(x_0)\,\mathrm{d}x \equiv 0 \end{aligned} \tag{3.16}$$

if the function $b_1(x_0)$ is properly normalized. But this can be achieved by an appropriate choice of $f(x_0)$ appearing in (3.10). Therefore the orthonormality relations

$$\langle \overline{\mathfrak{T}(j,k_1' \ldots k_n')}\,|\,\mathfrak{T}(j\,k_1 \ldots k_m)\rangle_W = \delta_{nm}\,\frac{1}{n!}\,p \sum_{\lambda_1 \ldots \lambda_n} (-1)^p\,\delta(k_1'-k_{\lambda_1}) \ldots \delta(k_n'-k_{\lambda_n}) \tag{3.17}$$

follow directly from (3.15) by observing (3.16).

4. Localized Free Fields

Besides the free particle states with definite four momentum, also a description of localized wave packets is needed for the purpose of S-matrix construction. Dividing the quantum numbers k of the maximal set of Sect. 2 into the four momentum quantum numbers $p := (\mathfrak{p}, p_0)$ and the additional quantum numbers s_a like spin, charge etc. an explicit representation of the corresponding states (with definite four momentum) is given by

$$|\,k_1 \ldots k_n\rangle := a^+(\mathfrak{p}_1\,s_1\,m) \ldots a^+(\mathfrak{p}_n\,s_n\,m)\,|\,0\rangle \tag{4.1}$$

where $a^+(\mathfrak{p}, s, m)$ are the creation operators of particles with these quantum numbers. To construct wave packets, we introduce the creation operators of localized particles by

$$q_{ns}^+(t) := \int \tilde{f}_n(\mathfrak{k}, t)\,a^+(\mathfrak{k}\,s\,m)\,\mathrm{d}\mathfrak{k} \tag{4.2}$$

where n denotes the set of wave packet functions and the $\tilde{f}_n$ are orthonormalized we obtain from (4.2) by observing the anticommutation relations of the a and a^+

$$[q_{ns}(t), q^+_{ms'}(t)]_+ = \delta_{nm}\,\delta_{ss'}, \quad [q_{ns}(t)\, q_{ms'}(t)]_+ = [q^+_{ns}(t)\, q^+_{ms'}(t)]_+ = 0. \tag{4.3}$$

Therefore the states of localized particles

$$|n_1 s_1 \dots n_m s_m t\rangle := q^+_{n_1 s_1}(t) \dots q^+_{n_n s_n}(t)\,|0\rangle \tag{4.4}$$

are orthonormalized due to (4.3)

$$\langle n_1' s_1' \dots n_m' s_m' t\,|\,n_1 s_1 \dots n_n s_n t\rangle = \delta_{mn}\, p \sum_{\lambda_1 \dots \lambda_n} (-1)^p\, \delta_{n_1' n \lambda_1}\, \delta_{s_1' s \lambda_1} \dots \delta_{n_m' n \lambda_m}\, \delta_{s'_{\lambda m} s_m} \,^*. \tag{4.5}$$

These properties can be transferred into functional space. Observing

$$|n_1 s_1 \dots n_n s_n t\rangle = \int \tilde{f}_{n_1}(\mathfrak{k}_1 t) \dots \tilde{f}_{n_n}(\mathfrak{k}_n t)\, a^+(\mathfrak{k}_1 s_1 m) \dots a^+(\mathfrak{k}_n s_n m)\,|0\rangle\, dk_1 \quad dk_n \tag{4.6}$$

and the definition of the corresponding functional states (1.6) we obtain by combination of (4.6) and (1.6) the localized state functionals

$$|\mathfrak{T}(j, n_1 s_1 \dots n_n s_n t)\rangle := \int \tilde{f}_{n_1}(\mathfrak{k}_1 t) \dots \tilde{f}_{n_n}(\mathfrak{k}_n t)\,|\mathfrak{T}(j, \mathfrak{k}_1 s_1 \dots \mathfrak{k}_n s_n)\rangle\, dk_1 \dots dk_n\,. \tag{4.7}$$

By $k := (\mathfrak{k}, s)$ the orthonormality relation (3.17) can be specified, to give the orthonormality relation

$$\langle\overline{\mathfrak{T}(j, n_1' s_1' \dots n_m' s_m' t)}\,|\,\mathfrak{T}(j, n_1 s_1 \dots n_n s_n t)\rangle = \delta_{mn}\, p \sum_{\lambda_1 \dots \lambda_n} (-1)^p\, \delta_{n_1' n \lambda_1}\, \delta_{s_1' s,_1} \dots \delta_{n_m' n \lambda_m}\, \delta_{s_m' s \lambda_m} \tag{4.8}$$

for the localized state functionals.

It is still convenient to express the localized state functionals (4.7) not in terms of the original field operators $\Psi_\alpha(x)$ but in terms of quasilocalized field operators arising from the definition (4.2). To do this we represent the creation (and destruction) operators by the original field operators. This gives

$$a(\mathfrak{p}, r) = (2\,\pi)^{-3/2} \int e^{ipx} f_r^\alpha(\mathfrak{p})\, \Psi_\alpha(x)\, d\mathfrak{r} \tag{4.9}$$

and it follows

$$q_{nr}(t) = \int f_{nr}^\alpha(\mathfrak{r})\, \Psi_\alpha(x)\, d\mathfrak{r} \tag{4.10}$$

with $\int f^n_{\alpha r}(\mathfrak{r})\, f^\alpha_{n'r'}(\mathfrak{r})\, d\mathfrak{r} = \delta_{nn'}\, \delta_{rr'}\,.$ (4.11)

Assuming an expansion of $j^\alpha(x)$ into

$$j^\alpha(x) = \sum_{ns} f^\alpha_{ns}(\mathfrak{r})\, j_{ns}(t) \tag{4.12}$$

we obtain by substitution of (4.12) into (1.6) by (4.10)

$$\mathfrak{T}(j, n_1 s_1 \dots n_n s_n t) := \langle 0\,|\,T \exp\{i \sum_{ns} \int q_{ns}(t)\, j_{ns}(t)\, dt\}\,|\,n_1 s_1 \dots n_n s_n t\rangle\,. \tag{4.13}$$

which gives the desired representation by quasilocal field operators.

* Aus satztechnischen Gründen mußte hier und im folgenden im Index statt λ_1 usw. $\lambda 1$ gesetzt werden.

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Appendix I

a) Quantum Mechanical Case

The ordinary Dirac equation reads

$$(-i\,\gamma^\mu_{\alpha\beta}\,\partial_\mu + m\,\delta_{\alpha\beta})\,\psi_\beta(x) = 0\,. \tag{I.1}$$

Defining the adjoint spinor by $\overline{\psi}_\beta(x) := \psi^\times_\alpha(x)\,\gamma^0_{\alpha\beta}$ its equation reads

$$\overline{\psi}_\alpha(x)\,(i\,\gamma^\mu_{\alpha\beta}\,\partial_\mu + m\,\delta_{\alpha\beta}) = 0\,. \tag{I.2}$$

Putting now

$$\psi_\beta(x) =: \varphi_{1\beta}(x)\,; \qquad \psi^\times_\beta(x) =: \varphi_{2\beta}(x) \tag{I.3}$$

real spinorial wave functions

$$\Psi_{\varrho\beta}(x) = c^{-1}_{\varrho\lambda}\,\varphi_{\lambda\beta}(x) \tag{I.4}$$

are introduced with

$$c^{-1} := (2)^{-\frac{1}{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad c := (2)^{-\frac{1}{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{I.5}$$

By elementary calculus Eqs. (I.1), (I.2) can be combined to give an equation for the real spinorial wave functions

$$(i\, G^{\mu}_{\alpha\varepsilon\beta\varrho}\, \partial_\mu + m\, \delta_{\alpha\varepsilon\beta\varrho})\, \Psi_{\beta\varrho}(x) = 0 \qquad (I.6)$$

with
$$G^{\alpha}_{\alpha\varepsilon\beta\varrho} := c^{-1}_{\varepsilon\varkappa}\, \Gamma^{\mu}_{\alpha\varkappa\beta\lambda}\, c_{\lambda\varrho} \qquad (I.7)$$

and
$$\Gamma^{\mu}_{\alpha\varkappa,\beta\lambda} := \begin{pmatrix} -\gamma^{\mu}_{\alpha\beta} & 0 \\ 0 & \gamma^{\mu\times}_{\alpha\beta} \end{pmatrix}. \qquad (I.8)$$

The transformation law of ordinary Dirac spinors reads

$$\psi'_\alpha(x') = S_{\alpha\beta}(a^{-1})\; \psi_\beta(a\,x' + L) \qquad (I.9)$$

for Poincaré transformations (a, L), Putting

$$S = \mathrm{Re}\, S + i\, \mathrm{Im}\, S$$

for real spinors the transformation law

$$\Psi'_{\mu\alpha}(x') = D_{\mu\alpha\varrho\beta}(a^{-1})\; \Psi_{\varrho\beta}(a\,x' + L) \qquad (I.10)$$

can be derived with

$$D_{\mu\alpha\varrho\beta} := \begin{pmatrix} \mathrm{Re}\, S_{\alpha\beta} & -\mathrm{Im}\, S_{\alpha\beta} \\ \mathrm{Im}\, S_{\alpha\beta} & \mathrm{Re}\, S_{\alpha\beta} \end{pmatrix}. \qquad (I.11)$$

As (I.1) and (I.2) are forminvariant equations under Poincaré transformations the derivation of (I.6) is valid in any frame of reference, and therefore (I.6) is also a forminvariant equation. Hence it follows by the usual procedure, that the G^μ transform acording to the transformation law

$$D^{-1}_{\chi\alpha\varrho\beta}\, G^{\lambda}_{\varrho\beta\varkappa\delta}\, D_{\varkappa\delta\varepsilon\nu} = a^{\lambda}_{\mu}\, G^{\mu}_{\chi\alpha\varepsilon\nu} \qquad (I.12)$$

(I.12) suggests to consider the G^μ matrices to be mixed co-contravariant tensors in spinor space due to their transformation properties. Therefore in analogy to ordinary tensor calculus we define the mixed spin tensors

$$\begin{aligned} {}^{\lambda}G^{\varkappa\delta}_{\varrho\beta} &:= G^{\lambda}_{\varrho\beta,\varkappa\delta} \quad \text{numerical identity} \\ D^{\varepsilon\nu}_{\varkappa\delta} &:= D_{\varkappa\delta,\,\varepsilon\nu}\,. \end{aligned} \qquad (I.13)$$

Then the Eq. (I.6) reads

$$(i^{\mu}\, G^{\beta\varrho}_{\alpha\varepsilon}\, \partial_\mu + m\, \delta^{\beta\varrho}_{\alpha\varepsilon})\; \Psi_{\beta\varrho}(x) = 0 \qquad (I.14)$$

and the transformation law (I.12)

$$D^{-1}{}^{\varrho\beta}_{\chi\alpha}\, D^{\varepsilon\nu}_{\varkappa\delta}\; {}^{\lambda}G^{\varkappa\delta}_{\varrho\beta} = a^{\lambda}_{\mu}\; {}^{\mu}G^{\varepsilon\nu}_{\chi\alpha} \qquad (I.15)$$

where D^{-1} is defined by the invariant relation

$$D^{-1}{}^{\varrho\beta}_{\chi\alpha}\, D^{\varkappa\alpha}_{\varkappa\delta} = \delta^{\varrho\beta}_{\varkappa\delta}\,. \qquad (I.16)$$

In the following besides the tensors (I.13) an additional tensor

$${}^{5}G^{\beta\varrho}_{\alpha\varepsilon} := c^{-1}{}^{\varkappa}_{\varepsilon}\; {}^{5}\Gamma^{\beta\lambda}_{\alpha\varkappa}\, c^{\varrho}_{\lambda} \qquad (I.17)$$

is required with

$${}^{5}\Gamma^{\lambda\beta}_{\alpha\varkappa} := \begin{pmatrix} -\delta^{\beta}_{\alpha} & 0 \\ 0 & \delta^{\beta}_{\alpha} \end{pmatrix}. \qquad (I.18)$$

By direct calculation it can be shows that (I.17) is a forminvariant quantity i. e.

$${}^{5}G^{\varepsilon\nu}_{\chi\alpha} = D^{-1}{}^{\varrho\beta}_{\chi\alpha}\, D^{\varepsilon\nu}_{\varkappa\delta}\; {}^{5}G^{\varkappa\delta}_{\varrho\beta}\,. \qquad (I.19)$$

Now we are in the position to consider the most important Poincaré invariant resp. covariant expressions, first of all the invariant $\bar\psi\psi$. By means of (I.4) we obtain

$$\bar\psi\,\psi := \psi^{\times}_{\alpha}\, \gamma^{0}_{\alpha\beta}\, \psi_\beta = \tfrac{1}{2}\, \Psi_{\mu\alpha}\, \delta_{\mu\varrho}\, \gamma^{0}_{\alpha\beta}\, \Psi_{\varrho\beta}\,. \qquad (I.20)$$

As the wavefunctions transform according to (I.10), and (I.20) has to be an invariant, the identity

$$\Psi_{\mu\alpha}\, \delta_{\mu\varrho}\, \gamma^{0}_{\alpha\beta}\, \Psi_{\varrho\beta} = \Psi_{\mu\alpha}\, g^{\mu\alpha\,\varrho\beta}\, \Psi_{\varrho\beta} \qquad (I.21)$$

has to be valid, where the metrical fundamental tensor in spinor space is defined by

$$g^{\mu\alpha\,\varrho\beta} := \delta_{\mu\varrho}\, \gamma^{0}_{\alpha\beta} \quad \text{numerical identity}\,. \qquad (I.22)$$

By means of the transformation properties of γ^0 it can be shown that g is a forminvariant quantity under Poincaré transformations, i. e.

$$g^{\mu\alpha\,\varrho\beta} = D^{\mu\alpha}_{\mu'\alpha'}\; D^{\varrho\beta}_{\varrho'\beta'}\; g^{\mu'\alpha'\,\varrho'\beta'} \qquad (I.23)$$

in accordance with the transformation law of a double contravariant tensor. The simultaneous validity of (I.23) and (I.15) both containing γ^0 is no contradiction, as the same phenomenon occurs in ordinary Dirac theory.

The metrical fundamental tensor can be used to raise resp. to lower the indices. Especially

$$\Psi^{\varrho\beta} = \Psi_{\mu\alpha}\, g^{\mu\alpha\,\varrho\beta} \qquad (I.24)$$

transforms like

$$\Psi'^{\varrho\beta}(x') = D^{-1}{}^{\nu\beta}_{\varkappa\lambda}\; \Psi^{\varkappa\lambda}(a\,x' + L) \qquad (I.25)$$

due to the invariance of

$$\Psi_{\mu\alpha}\, g^{\mu\alpha\,\varrho\beta}\, \Psi_{\varrho\beta} = \Psi^{\varrho\beta}\, \Psi_{\varrho\beta}\,. \qquad (I.26)$$

The next important quantity is the current. It is defined in ordinary theory by $j^\lambda = \bar\psi\gamma^\lambda\psi$. For our purposes we require only a specialized form namely

$$j^\lambda(x\,|\,k\,k') := \psi^{x}_{\alpha}(x\,|\,k)\, (\gamma_0\,\gamma^\lambda)_{\alpha\beta}\, \psi_\beta(x\,|\,k') \qquad (I.27)$$

where $\psi_\alpha(x\,|\,k)$ is a solution of (I.1) for the maximal set of quantum numbers, defined in Sect. 2. For (I.27) a conservation law

$$\int_{\Sigma} d\sigma_\lambda\, j^\lambda(x\,|\,k\,k') = \int_{\Sigma'} d\sigma_\lambda\, j^\lambda(x\,|\,k\,k') \qquad (I.28)$$

is valid, with Σ and Σ' spacelike surfaces. Specializing to planes

$$\Sigma := a^{\mu}\, x_\mu + b = 0; \quad \Sigma' := a'^{\mu}\, x_\mu + b' = 0. \qquad (I.29)$$

(I.29) can be written

$$\int a_\lambda\, j^\lambda(x\,|\,k\,k')\;\delta(a^\mu x_\mu+b)\;\mathrm{d}x \;=\int a\, j^\lambda(x\,|\,k\,k')\;\delta(a'^\mu x_\mu+b')\;\mathrm{d}x\,. \tag{I.30}$$

As b can varied arbitrarily we may have with $\int G(b)\;\mathrm{d}b=1$

$$\begin{aligned}\int a_\lambda\, j^\lambda(x\,|\,k\,k')\;\delta(a^\mu x_\mu+b)\;\mathrm{d}x &= \int a_\lambda\, j^\lambda(x\,|\,k\,k')\;\delta(a^\mu x_\mu+b)\;G(b)\;\mathrm{d}x\,\mathrm{d}b\\ &=\int a_\lambda\, j^\lambda(x\,|\,k\,k')\;G(-a^\mu x_\mu)\;\mathrm{d}x\,.\end{aligned} \tag{I.31}$$

On the other hand by putting $a'^\mu=(1,0,0,0)$ we obtain combining (I.31), (I.30)

$$\int a_\lambda\, j^\lambda(x\,|\,k\,k')\;G(-a^\mu x_\mu)\;\mathrm{d}x=\int \psi_\alpha^\times(\mathfrak{r}\,0\,|\,k)\;\psi_\alpha(\mathfrak{r}\,0\,|\,k')\;\mathrm{d}\mathfrak{r}\;=\delta_{kk'} \tag{I.32}$$

i. e.

$$\int \psi_\alpha^\times(x\,|\,k)\;(\gamma_0\,\gamma^\lambda)_{\alpha\beta}\,a_\lambda\,\delta(x-y)\;G(-a^\mu x_\mu)\;\psi_\beta(y\,|\,k')\,\mathrm{d}y\;\mathrm{d}y\;=\delta_{kk'}\,. \tag{I.33}$$

Now we express this relation in real spinors. We state that

$$\int f^{\alpha\varkappa}(x\,|\,k)\;({}^5G^\lambda\,G)^{\varrho\beta}_{\alpha\varkappa}\,a_\lambda\,\delta(x-y)\;G(-a^\mu x_\mu)\;\;f_{\varrho\beta}(y\,|\,k')\;\mathrm{d}x\,\mathrm{d}y=\delta_{kk'} \tag{I.34}$$

is valid with

$$f_{\varrho\beta}(x\,|\,k)\;:=\;c_\varrho^\lambda\,\varphi_{\lambda\beta}(x\,|\,k)\,. \tag{I.35}$$

To prove this statement we observe (I.34) to be an invariant under Poincaré transformations. Therefore (I.34) is valid, if it is valid in a special frame of reference. For this special frame we choose

$$G(-a^\mu x_\mu)\;\equiv\;\delta(x_0)\,. \tag{I.36}$$

Then (I.34) becomes

$$\int f^{\alpha\varkappa}(\mathfrak{r},0\,|\,k)\;({}^5G^0\,G)^{\varrho\beta}_{\alpha\varkappa}\,f_{\varrho\beta}(\mathfrak{r},0\,|\,k')\;\mathrm{d}\mathfrak{r}\;=\int[f_{1\alpha}(\mathfrak{r},0\,|\,k)\;f_{1\alpha}(\mathfrak{r},0\,|\,k')\;+f_{2\alpha}(\mathfrak{r},0\,|\,k)\;f_{2\alpha}(\mathfrak{r},0\,|\,k')]\;\mathrm{d}\mathfrak{r} \tag{I.37}$$

by straightforward calculation. On the other hand we have by direct substitution

$$\delta_{kk'}=\int \psi_\alpha^\times(\mathfrak{r},0\,|\,k)\;\psi_\beta(\mathfrak{r},0\,|\,k')\;\mathrm{d}\mathfrak{r}\;=\int[f_{1\alpha}\,f'_{1\alpha}+f_{2\alpha}\,f'_{2\alpha}]\;\mathrm{d}\mathfrak{r}+i\int[f_{1\alpha}\,f'_{2\alpha}\;-f_{2\alpha}\,f'_{1\alpha}]\;\mathrm{d}\mathfrak{r}\,. \tag{I.38}$$

As f is completely real it follows

$$\int[f_{1\alpha}(\mathfrak{r},0\,|\,k)\;f_{1\alpha}(\mathfrak{r},0\,|\,k')\;+f_{2\alpha}(\mathfrak{r},0\,|\,k)\;f_{2\alpha}(\mathfrak{r},0\,|\,k')\;\mathrm{d}\mathfrak{r}=\delta_{kk'} \tag{I.39}$$

and

$$\int[f_{1\alpha}(\mathfrak{r},0\,|\,k)\;f_{2\alpha}(\mathfrak{r},0\,|\,k')\;-f_{2\alpha}(\mathfrak{r},0\,|\,k)\;f_{1\alpha}(\mathfrak{r},0\,|\,k')\;\mathrm{d}\mathfrak{r}\equiv 0\,. \tag{I.40}$$

Combining (I.37) with (I.40) the statement (I.34) is proven.

b) Field Quantized Case

In this case $\psi(x)$ becomes an operator in Hilbert space, and the adjoint operator $\overline{\psi}(x)$ is defined by $\overline{\psi}_\beta(x)=\psi^+(x)\,\gamma$ where ψ^+ means the Hermitean conjugate of ψ in Hilbert space. These operators satisfy the Eqs. (I.1) and (I.2). Additionally they satisfy the anticommutation relations

$$[\psi_\alpha(x)\;\psi_\beta^+(x')]_{+/x_0=x_0'}=\delta(\mathfrak{r}-\mathfrak{r}')\;\delta_{\alpha\beta}\,\mathbf{1}\,. \tag{I.41}$$

$$[\psi_\alpha(x)\;\psi_\beta(x')]_{+/x_0=x_0'}\;=[\psi_\alpha^+(x)\;\psi_\beta^+(x')]_{+/x_0=x_0'}=0\,. \tag{I.42}$$

Defining now

$$\psi_\beta(x)\;=:\varphi_{1\beta}(x)\,;\qquad \psi_\beta^+(x)\;=:\varphi_{2\beta}(x)\,. \tag{I.43}$$

Hermitean field operators are introduced by

$$\Psi_{\varrho\beta}(x)=c_{\varrho\lambda}^{-1}\,\varphi_{\lambda\beta}(x)\,. \tag{I.44}$$

Then the anticommutation rules (I.41), (I.42) go over into

$$[\Psi_{\mu\alpha}(x)\;\Psi_{\varrho\beta}(x')]_{+/x_0=x_0'}=\delta(\mathfrak{r}-\mathfrak{r}')\;\delta_{\mu\alpha\,\varrho\beta}\,\mathbf{1} \tag{I.45}$$

and the Hermitean operators satisfy Eq. (1.14).

The operator transformation law is for ordinary Dirac operators

$$U\,\psi_\beta(x')\;U^{-1}=S_{\beta\alpha}(a^{-1})\;\psi_\alpha(a\,x'+L) \tag{I.46}$$

and in Hermitean operators

$$U\,\Psi_{\beta\varrho}(x')\;U^{-1}=D^{\varkappa\lambda}_{\beta\varrho}\,\Psi_{\varkappa\lambda}(a\,x'+L) \tag{I.47}$$

For simplicity we replace the double indices β_ϱ by only one index in the following calculations as long as no misunderstanding is possible. Then we obtain the Eqs. (1.1), (1.2), (1.3).

Appendix II

Formally we introduced anticommuting sources $j_{m\alpha}(x)$ and corresponding functional derivatives

$\partial_{m\alpha}(x)$ for Hermitean field operators defined in (I.44). Like in Appendix I we replace the double indices $n\,\alpha$ by only one index in the following calculations as long as no misunderstanding is possible. Before discussing the existence of these operators we write down the conditions which they should satisfy. These are the Poincaré invariant anticommutation rules

$$[j_\alpha(x)\, j_{\alpha'}(x')]_+ = 0\,, \quad [j_\alpha(x)\, \partial^{\alpha'}(x')]_+ = \delta_\alpha^{\alpha'}\, \delta(x-x')\,, \quad [\partial^\alpha(x)\, \partial^{\alpha'}(x')]_+ = 0 \tag{II.1}$$

together with the transformation law

$$V\, j_\alpha(x)\, V^{-1} = D_\alpha^\beta\, j_\beta(a\,x+b)\,, \quad V\, \partial^\alpha(x)\, V^{-1} = D^{-1}{}^\alpha_\beta\, \partial^\beta(a\,x+b) \tag{II.2}$$

for Poincaré transformation (a, b) where V is a representation in functional space, and

$$\partial^\alpha(x) := g^{\alpha\beta}\, \partial_\beta(x)\,; \quad j^\alpha(x) := g^{\alpha\beta}\, j_\beta(x) \tag{II.3}$$

are defined in accordance with (I.24). It can be shown easily that (II.2) effects the invariance of (II.1). Therefore (II.2) and (II.1) are consistent conditions.

To prove the existence of j and ∂ we show that this problem can be reduced to the existence problem of ordinary creation and destruction operators of free relativistic fields. To do this we make the ansatz

$$j_\alpha(x) = \sum_\varkappa \int c_\alpha(p\,|\,\varkappa)\, e^{ipx}\, a_\varkappa^+(p)\, dp\,, \tag{II.4}$$

where the integration runs over the entire Lorentz space. (II.4) can be divided into invariant subspace integrations

$$j_\alpha(x) = \sum_\varkappa \Big[\int_{Lt} c_\alpha^t(p\,|\,\varkappa)\, e^{ipx}\, a_\varkappa^{t+}(p)\, dp + \int_{Ls} c_\alpha^s(p\,|\,\varkappa)\, e^{ipx}\, a_\varkappa^{s+}(p)\, dp\Big]\,, \tag{II.5}$$

where L_t denotes the manyfold of timelike or lightlike vectors $p^2 \geqq 0$ and L_s the spacelike vectors with $p^2 < 0$. To define the quantities occuring in (II.5) we observe that a division into classes is possible by the mass values $p^2 = m^2$ (timelike, lightlike), $p^2 = -\mu^2$ (spacelike). Then a class is defined by a fixed value of m esp. μ, and each p can be contained only in one class. In each class exists a rest system with $\boldsymbol{m} := m\, \mathfrak{e}_0$ resp. $\boldsymbol{\mu} := \mu\, \mathfrak{e}_1$ and for any p of this class a Lorentz transformation $a^t p = \boldsymbol{m}$ resp. $a^s p = \boldsymbol{\mu}$ exists. Therefore we may define

$$c_\alpha^t(p\,|\,\varkappa) := D_\alpha^\beta\, (a^{t-1})\, c_\beta^t(\boldsymbol{m}\,|\,\varkappa)\,, \quad p \in C_m\,, \qquad c_\alpha^s(p\,|\,\varkappa) := D_\alpha^\beta\, (a^{s-1})\, c_\beta^s(\boldsymbol{\mu}\,|\,\varkappa)\,, \quad p \in C_\mu\,. \tag{II.6}$$

Additionally one should distinguish between forward and backward lightcone, but this is a complication which can be omitted for these considerations.

As the scalarproduct is an invariant quantity the definition of classes is Lorentz invariant, too. Therefore for any Lorentz transformation $w\,p' = p$, with $p \in C_m$ resp. C_μ it follows $p' \in C_m$ resp. C_μ, and the transformation law

$$c'_\alpha(p'\,|\,\varkappa) = D_\alpha^\beta\, (w^{-1})\, c_\beta(w\,p'\,|\,\varkappa) \tag{II.7}$$

is consistent with the definitions (II.6). To investigate the transformation law of (II.4) we calculate

$$D_\alpha^\beta\, (w^{-1})\, j_\beta(w\,x) = \cdot \sum_\varkappa \int D_\alpha^\beta\, (w^{-1})\, c_\beta(p\,|\,\varkappa)\, e^{ipwx}\, a_\varkappa^+(p)\, dp\,, \tag{II.8}$$

which becomes with $p\,w = p'$, (II.7) and the group properties of D

$$D_\alpha^\beta\, (w^{-1})\, j_\beta(w\,x) = \sum_\varkappa \int c_\alpha(p'\,|\,\varkappa)\, e^{ip'x}\, a_\varkappa^+(p'\,w^{-1})\, dp'. \tag{II.9}$$

Due to the arguments given before also the $a_\varkappa(p)$ belong uniquely to one class. Therefore we put

$$a_\varkappa^{t+}(p) := a_\varkappa^+(p, m)\,, \quad p \in C_m\,, \qquad a_\varkappa^{s+}(p) := a_\varkappa^+(p, \mu)\,, \quad p \in c_\mu\,, \tag{II.10}$$

where $a\;(p, m)$ resp. $a\;(p, \mu)$ are creation operators of ordinary free relativistic fields of mass m, resp. $i\,\mu$. For these operators the transformation law

$$V(w)\, a_\varkappa^+(p'\,w^{-1})\, V^{-1}(w) = a_\varkappa^+(p') \tag{II.11}$$

is valid, and therefore substitution of (II.11) into (II.9) gives (II.2) for homogeneous Lorentz transformations. To construct the functional derivatives, we put

$$\partial^\beta(x) = \sum_\varkappa \int c^\beta(p\,|\,\varkappa)^\times\, e^{-ipx}\, a_\varkappa(p)\, dp\,. \tag{II.12}$$

and require

$$\sum_\varkappa c_\beta(\boldsymbol{m}\,|\,\varkappa)\, c^{\beta'}(\boldsymbol{m}\,|\,\varkappa)^x = \delta_\beta^{\beta'}\,. \tag{II.13}$$

Then by the invariance of

$$\sum_\varkappa c_\beta(p\,|\,\varkappa)\, c^{\beta'}(p\,|\,\varkappa)^x = \sum_\varkappa c_\beta(\boldsymbol{m}\,|\,\varkappa)\, c^{\beta'}(\boldsymbol{m}\,|\,\varkappa)^x \tag{II.14}$$

one realizes by direct calculation that (II.1) is satisfied. It is easy to extend (II.9) to inhomogeneous Lorentz transformations, so that an explicit representation of j and ∂ is given, satisfying (II.1), (II.2), provided the $a_\varkappa(p, m)$ resp. $a_\varkappa(p, \mu)$ exist. This is asumed in the following.

To construct appropriate functional spaces we do need an additional information about the functional ground state. In the simplest case we require

$$\partial_\beta(x) \mid \varphi_0\rangle = 0\,, \tag{II.15}$$

where $|\varphi_0\rangle$ denotes the functional ground state. The possibilty of the condition (II.15) is obvious due to (II.12). Starting from this formula various functional spaces may be generated by using the weighting functional

$$e^{-jGj} := \exp\{-\int j_\alpha(x)\, G^{\alpha\beta}(x,y)\, j_\beta(y)\, \mathrm{d}x\, \mathrm{d}y\}\,, \tag{II.16}$$

where G is an antisymmetric function in (α, x) and (β, y). For (II.16) the relation

$$A^\varkappa(x)\, e^{-jGj} = e^{-jGj}\, \partial^\varkappa(x) \tag{II.17}$$

holds with

$$A^\varkappa(x) := [2\int G^{\varkappa\beta}(x,y)\, j_\beta(y)\, \mathrm{d}y + \partial^\varkappa(x)]\,. \tag{II.18}$$

Then a weighted ground state $|\varphi_0\rangle_{\mathrm{G}}$ may be defined by

$$|\varphi_0\rangle_{\mathrm{G}} := e^{-jGj}\, |\varphi_0\rangle \tag{II.19}$$

and from (II.15) (II.17) follows

$$A^\varkappa(x) \mid \varphi_0\rangle_{\mathrm{G}} = 0\,. \tag{II.20}$$

The total Hermitean conjugate, i. e. the Hermitean conjugate with respect to spinorial as well as to functional space of (II.18) reads

$$A^{\varkappa +}(x) = [2\int \partial_\beta(y)\, G^{\varkappa\beta}(x,y)^+\, \mathrm{d}y + j^\varkappa(x)] \tag{II.21}$$

by observing $\partial_\varkappa(x) = j_\varkappa^+(x)$ due to (II.4), (II.12). Then the Hermitean conjugate of (II.20) reads

$$(A^\varkappa(x) \mid \varphi_0\rangle_{\mathrm{G}})^+ = {}_{\mathrm{G}}\langle\varphi_0 \mid A^{\varkappa +}(x) = 0\,. \tag{II.22}$$

The results concerning this new set of operators may be summarized in the following formulas, in which raising and lowering of indices is performed according to (II.3). First the definition

$$A^\varkappa(x) := [2\int G^{\varkappa\beta}(x,y)\, j_\beta(y)\, \mathrm{d}y + \partial^\varkappa(x)]\,,$$
$$A_\varkappa^+(x) := [2\int G_{\varkappa\beta}^x(x,y)\, \partial^\beta(y)\, \mathrm{d}y + j_\varkappa(x)] \tag{II.23}$$

satisfying the commutation rules

$$[A^\varkappa(x)\, A^{\varkappa'}(x')]_+ = 0\,,$$
$$[A^\varkappa(x)\, A_{\varkappa'}^+(x')]_+ = a_{\varkappa'}^{\varkappa}(x, x')\,, \tag{II.24}$$
$$[A_\varkappa^+(x)\, A_{\varkappa'}^+(x')]_+ = 0$$

with

$$a_{\varkappa'}^{\varkappa}(x\,x') := 4\int G^{\varkappa\beta}(x,y)\, G_{\beta\varkappa'}^x(y\,x')\, \mathrm{d}y + \delta_{\varkappa'}^{\varkappa}\, \delta(x-x')\,, \tag{II.25}$$

where (II.25) is a symmetric function with respect to $(\varkappa\, x)$ and $(\varkappa'\, x')$. Further the transformation law

$$V A^\varkappa(x)\, V^{-1} = D^{-1\varkappa}_{\ \ \varrho} A^\varrho(a\,x+b)\,,$$
$$V A_\varkappa^+(x)\, V^{-1} = D_\varkappa^\varrho\, A_\varrho^+(a\,x+b) \tag{II.26}$$

and the relations

$$A^\varkappa(x) \mid \varphi_0\rangle_{\mathrm{G}} = {}_{\mathrm{G}}\langle\varphi_0 \mid A_\varkappa^+(x) = 0\,. \tag{II.27}$$

By means of (II.23) to (II.27) a basis vector system may be defined, namely the Hermitean functionals for anticommuting sources and weighted ground state.

$$|I_n(\underset{\alpha_1}{z_1} \ldots \underset{\alpha_n}{z_n})\rangle := \frac{1}{\sqrt{n!}}\, A_{\alpha_1}^+(z_1) \ldots A_{\alpha_n}^+(z_n)\, |\varphi_0\rangle_{\mathrm{G}}\,,$$
$$n = 0, 1, \ldots, \infty \tag{II.28}$$

and their Hermitean conjugate

$$\langle I_n(\underset{\alpha_1}{z_1} \ldots \underset{\alpha_n}{z_n})| = \frac{1}{\sqrt{n!}}\, {}_{\mathrm{G}}\langle\varphi_0 \mid A_{\alpha_1}(z_1) \ldots A_{\alpha_n}(z_n)\,. \tag{II.29}$$

For the formation of a functional scalar product the adjoint state

$$\overline{\langle I_n(\underset{\alpha_1}{z_1} \ldots \underset{\alpha_n}{z_n})|} := \langle I_n(\underset{\beta_1}{z_1} \ldots \underset{\beta_n}{z_n})|\, ({}^5G\,{}^0G)_{\alpha_1'}^{\beta_1} ({}^5G\,{}^0G)_{\alpha_n'}^{\beta_k} \tag{II.30}$$

is required. Then by means of (II.23) to (II.27) the functional scalar product

$$\overline{\langle I_n(\underset{\alpha_1'}{z_1'} \ldots \underset{\alpha_m'}{z_m'})|}\, I_n(\underset{\alpha_1}{z_1} \ldots \underset{\alpha_n}{z_n})\rangle \tag{II.31}$$
$$= \frac{1}{n!}\, \delta_{nm}\, p \sum_{\lambda_1 \ldots \lambda_n} (-1)^p A(\underset{\alpha_1'\ \alpha_{\lambda 1}}{z_1' z_{\lambda 1}}) \ldots A(\underset{\alpha_n'\ \alpha_{\lambda n}}{z_n' z_{\lambda n}})\, \|\varphi_{0\mathrm{G}}\|$$

can be evaluated with

$$A(\underset{\alpha\ \alpha'}{z\, z'}) := ({}^5G\,{}^0G)_\alpha^\beta\, a(\underset{\beta\ \alpha'}{z\, z'})\,. \tag{II.32}$$

As the norm of $|\varphi_0\rangle_{\mathrm{G}}$ is an unknown quantity it is convenient to define the renormalized scalar product by

$$\langle \overline{I_m} \mid I_n\rangle_r := \|\varphi_{0\mathrm{G}}\|^{-1}\, \langle \overline{I_m} \mid I_n\rangle\,. \tag{II.33}$$

In addition to the basic functionals (II.28) the non-orthogonal Dyson functionals are required. They are defined by

$$- \sum_{r=1}^{n} (-1)^r p' \sum_{\lambda_1 \lambda_3 \ldots \lambda_n = 1}^{n} \mp r (-1)^{p'} (-1)^2 \Theta(t_{\lambda_1} - t_r)\, \Theta(t_r - t_{\lambda_3}) \ldots \Theta(t_{\lambda_{n-1}} - t_{\lambda_n})\, x_{\lambda_1} x_{\lambda_3} \ldots x_{\lambda_n} \delta(x - x_r)$$

$$\vdots \qquad \text{(III.7)}$$

$$- \sum_{r=1}^{n} (-1)^r p' \sum_{\lambda_1 \ldots \lambda_{n-1} = 1}^{n} \mp r (-1)^{p'} (-1)^n \Theta(t_{\lambda_1} - t_{\lambda_2}) \ldots \Theta(t_{\lambda_{n-1}} - t_r)\, x_{\lambda_1} \ldots x_{\lambda_{n-1}} \delta(x - x_r)\,.$$

As the sign of $(-1)^{p'}$ is fixed by the orignal arrangement $\lambda_1 \ldots \lambda_n$, any change in the variables $\lambda_1 \to \lambda_2$, $\lambda_2 \to \lambda_3, \ldots$ is connected with a change in sign. Observing this, (II.7) can be rearranged to give

$$\frac{\partial}{\partial t} T x_1 \ldots x_n x_n = T x_1 \ldots x_n \dot{x}$$

$$+ \sum_{r=2}^{n} (-1)^r p \sum_{\lambda_2 \ldots \lambda_n = 1}^{n} \mp r (-1)^p x_{\lambda_1} \ldots x_{\lambda_{n-1}} \delta(x - x_r)\, \Theta(t_{\lambda_1} - t_{\lambda_2}) \ldots \Theta(t_{\lambda_{n-2}} - t_{\lambda_{n-1}})$$

$$\times \{\Theta(t_r - t_{\lambda_1}) + \vartheta(t_r - t_{\lambda_2}) - \vartheta(t_r - t_{\lambda_1}) + \ldots \Theta(t_{\lambda_{n-1}} - t_r)\} \qquad \text{(III.8)}$$

$$= T x_1 \ldots x_n x - \sum_{r=1}^{n} (-1)^r \delta(x - x_r)\, T x_1 \ldots x_{r-1} x_{r+1} \ldots x_n\,.$$

Combining (III.8) with the equation of motion (1.1) we obtain

$$(i^\mu G_\alpha^\beta \partial_\mu + m\, \delta_\alpha^\beta)\, T\, \Psi_{\beta_1}(x_1) \ldots \Psi_{\beta_n}(x_n)\, \Psi_\beta(x)$$

$$= -i\, {}^0G_\alpha^\beta \sum_{r=1}^{n} (-1)^r \delta(x - x_r)\, \delta_{\beta_r \beta}\, T\, \Psi_{\beta_1}(x_1) \ldots \Psi_{\beta_{r-1}}(x_{r-1})\, \Psi_{\beta_{r+1}}(x_{r+1}) \ldots \Psi_{\beta_n}(x_n)\,. \qquad \text{(III.9)}$$

Observing further

$$\partial(x)\, j(x_1) \ldots j(x_n) = \sum_n (-1)^{r+1} \delta(x_r - x)\, j(x_1) \ldots j(x_{r-1})\, j(x_{r+1}) \ldots j(x_n)$$

$$+ (-1)^n j(x_1) \ldots j(x_n)\, \partial(x)\,. \qquad \text{(III.10)}$$

The functional derivative of $|\mathfrak{T}(j)\rangle$ becomes with (II.15)

$$\partial_\beta(x)\, |\mathfrak{T}(j)\rangle = i \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \langle 0 | T\, \Psi_{\beta_1}(x_1) \ldots \Psi_{\beta_n}(x_n)\, \Psi_\beta(x)\, | a \rangle\, j^{\beta_1}(x_1) \ldots j^{\beta_n}(x_n)\, | \varphi_0 \rangle\, dx_1 \ldots dx_n\,. \qquad \text{(III.11)}$$

Combination of (II.1) with (II.9) then gives after elementary operations the desired functional equation

$$(i^\mu G_\alpha^\beta \partial_\mu + m\, \delta_\alpha^\beta)\, \partial_\beta(x)\, |\mathfrak{T}(j)\rangle = -i\, {}^0G_\alpha^\beta j_\beta(x)\, |\mathfrak{T}(j)\rangle\,. \qquad \text{(III.12)}$$